2014

MATHEMATICS

(Major)

Paper : 5.2

(Topology)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : $1\times7=7$

(a) Consider the set $\mathbb{R}$ with the usual metric. Which of the following sets are open in $\mathbb{R}$?

$$A=[0,1[,\ B=]0,1[,\ C=]0,1],\ D=[0,1],$$
$$E=]0,1[\cup[2,3[,\ F=\{1\},\ G=\{1,2,3\}$$

(b) Find the interior of the following subsets of $\mathbb{R}$ with respect to the usual metric :

$$A=]0,1[,\ B=[0,1[$$

(c) Give an example of a metric space which is not complete.

(d) Let $X=\{a, b, c\}$ and

$$\mathscr{T}=\{\phi, X, \{a\}, \{b\}, \{a, b\}\}$$

State whether $\mathscr{T}$ is a topology on X.

(e) Let $X = \{a, b, c, d, e\}$ and

$$\mathscr{B} = \{\{a, b\}, \{b, c\}, \{a, d, e\}\}$$

Find the topology on X generated by $\mathscr{B}$.

(f) Let $X = \{1, 2, 3, 4\}$ and

$$\mathscr{T} = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3, 4\}\}$$

Let $f : X \to X$ be defined by $f(1) = 2$, $f(2) = 4$, $f(3) = 2$, $f(4) = 3$. State whether f is continuous at 3.

(g) Define Banach space and give an example.

2. Answer the following questions : 2×4=8

(a) If (X, d) is any discrete metric space, describe the open spheres for (X, d).

(b) Let $X = \{a, b, c\}$ and

$$\mathscr{T} = \{\phi, X, \{b\}, \{a, c\}\}$$

Find the interior and the closure of the set $\{a, b\}$ in $(X, \mathscr{T})$.

(c) Show that in a normed linear space $(X, ||\cdot||)$

$$|\,||x|| - ||y||\,| \leq ||x - y|| \quad \forall x, y \in X$$

(d) Define an inner product on the space $\mathbb{C}^n$ and using this inner product, construct a norm on this space.

3. Answer the following questions : 5×3=15

(a) Prove that every convergent sequence in a metric space is a Cauchy sequence. Is the converse true? Justify.

(b) Let $(Y, \mathscr{T}_Y)$ be a subspace of the topological space $(X, \mathscr{T})$. Prove that a subset A of Y is closed in Y if and only if there exists a set F closed in X such that $A = F \cap Y$.

Or

Prove that a mapping f from a topological space X into another topological space Y is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for every set $A \subset X$.

(c) Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. Prove that for all $x, y \in X$

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$$

Or

In an inner product space $(X, \langle \cdot, \cdot \rangle)$, if $x_n \to x$ and $y_n \to y$, then show that

$$\langle x_n, y_n \rangle \to \langle x, y \rangle$$

4. Answer the following questions : 10×3=30

(a) Prove that every non-empty open set on the real line is the union of a countable class of pairwise disjoint open intervals. 10

Or

Let (X, d) be a metric space and $\{F_n\}$ be a nested sequence of non-empty closed subsets of X such that $\delta(F_n) \to 0$ as $n \to \infty$. Prove that X is complete if and only if $\bigcap_{n=1}^{\infty} F_n$ consists of exactly one point.

(b) Prove that a metric space is second countable if and only if it is separable. 10

Or

Define uniformly continuous mapping in metric spaces. Give an example to show that a continuous mapping need not be uniformly continuous. Prove that the image of a Cauchy sequence under a uniformly continuous mapping is itself a Cauchy sequence. 1+3+6=10

(c) Prove that a metric space is compact if and only if it is complete and totally bounded. 10

Or

Prove that the continuous image of a connected metric space is connected. Also, show that the range of a continuous real valued function defined on a connected space is an interval.

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