2014

MATHEMATICS

(Major)

Paper : 5.1

(Real and Complex Analysis)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×7=7

(a) State the Schwarz's theorem associated with the functions of several variables.

(b) Let f be a function defined on $[0,1]$ as follows :

$$f(x)=\begin{cases} c, \text{ positive constant}, & \text{when } x \neq 0 \\ 0, \text{ when } x=0 \end{cases}$$

Then f is Riemann integrable. State a reason without any calculation.

(c) For what values of p, the improper integral

$$\int_a^b \frac{dx}{(x-a)^p}$$

is convergent?

(d) Consider the following function f, defined as

$$f(x, y) = (x^2 - y)(2x^2 - y)$$

be such that $f_{xx}f_{yy} - (f_{xy})^2 = 0$ at (0, 0). Then what about the maxima and minima of the function f at (0, 0)?

(e) Define the harmonic conjugate of a function.

(f) Give an example of a conformal transformation.

(g) Evaluate

$$\oint_C \frac{dz}{z-a}$$

where C is a simple closed curve and $z = a$ is outside C.

2. Answer the following questions : 2×4=8

(a) Let f and g be Riemann integrables on $[a, b]$ and g keeps the same sign over $[a, b]$. Then show that there exists a number c lying between the bounds of f such that

$$\int_a^b fg\,dx = c\int_a^b g\,dx$$

(b) Prove that $\Gamma(m+1) = m\Gamma(m)$, $\forall\, m > 0$.

(c) Show that the function $e^x(\cos y + i\sin y)$ is holomorphic.

(d) Find the inverse of the point z with respect to the circle $|z| = r$.

3. Answer any *three* parts : 5×3=15

(a) Show that the function f, where

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

is continuous but not differentiable at the origin. 2½+2½=5

(b) Show that the function $[x]$, where $[x]$ denotes the greatest integer not greater than x, is Riemann integrable in [0, 3], and

$$\int_0^3 [x]\,dx = 3$$

3+2=5

(c) Let f and g be two positive functions in $[a, b]$ such that

$$\lim_{x \to a+} \frac{f(x)}{g(x)}$$

exists and is a non-zero finite number. Then show that the two integrals

$$\int_a^b f\,dx \text{ and } \int_a^b g\,dx$$

converge and diverge together at a. 5

(d) Show that an analytic function cannot have a constant modulus without reducing to a constant. 5

(e) What is meant by preservance of cross-ratio under bilinear transformation? Find the bilinear transformation which maps the points $z = -2, 0, 2$ into the points $w = 0, i, -i$. 2+3=5

4. Answer *either (a) or (b)* :

(a) *(i)* If f_x and f_y are both differentiable at a point (a, b) of the domain of definition of a function f, then prove that $f_{xy}(a, b) = f_{yx}(a, b)$. 6

(ii) Let V be a function of two variables x and y, and $x = r\cos\theta$, $y = r\sin\theta$. Then prove that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}$$ 4

(b) *(i)* Prove that the integral

$$\int_0^\infty x^{m-1} e^{-x} dx$$

is convergent if and only if $m > 0$. 6

(ii) Define absolutely convergent improper integral. Prove that every absolutely convergent integral is convergent. 1+3=4

5. Answer *either (a) or (b)* :

(a) Let f be a bounded and Riemann integrable function on $[a, b]$, and there exists a function F such that $F' = f$ on $[a, b]$. Then prove that

$$\int_a^b f\, dx = F(b) - F(a)$$

Hence show that

$$\int_0^1 f\, dx = \frac{2}{3}$$

where f is defined by

$$f(x) = \frac{1}{2^n}, \quad \text{when} \quad \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$$

$(n = 0, 1, 2, \cdots)$, $f(0) = 0$

is Riemann integrable on $[0, 1]$. 5+5=10

(b) Let f be bounded and Riemann integrable on $[a, b]$. Then prove that $|f|$ is also bounded and Riemann integrable on $[a, b]$. Also prove that

$$\left|\int_a^b f\, dx\right| \le \int_a^b |f|\, dx$$

Is the converse of the first part true? Justify. 4+2+4=10

6. Answer *either (a) or (b)* :

(a) *(i)* Show that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, \; z \neq 0 \text{ and } f(0) = 0$$

is continuous and that Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist. 6

(ii) Find the analytic function of which the real part is

$$e^x(x\cos y - y\sin y)$$ 4

(b) *(i)* Prove that every Möbius transformation maps circles or straight lines into circles or straight lines. 6

(ii) Define fixed points of a bilinear transformation. Under what conditions, a bilinear transformation has one or two finite fixed points? 1+3=4

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