Total No. of printed pages = 10

3 (Sem 4) MAT M1

2015

MATHEMATICS

(Major)

Theory Paper : M-4.1

(Real Analysis)

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : $1\times10=10$

(a) Give an example of a set which is not an interval but is a neighbourhood of each of its points.

(b) Define an open subset of real numbers.

[Turn over

(c) The set of limit points of $\{1, 3, 5, 7, 9\}$ is

(i) $\{1, 3\}$

(ii) $\{7, 9\}$

(iii) $\{1, 3, 5, 9\}$

(iv) None of these

(Choose the correct answer)

(d) Write whether the following statement is true or false :

A sequence having only one limit point is convergent.

(e) The sequence $\left\{\frac{(-1)^n}{n}\right\}$ is

(i) Convergent

(ii) Divergent

(iii) Oscillates finitely

(iv) Oscillates infinitely

(Choose the correct answer)

(f) Fill in the blank :

If a function f is derivable on a closed interval [a, b] and $f^1(a) < 0$ and $f^1(b) > 0$ then there exists at least one point c between a and b such that $f^1(c) =$ _____.

(g) If $f(x) = \begin{cases} x & , 0 < x < 1 \\ 3 - x, & 1 \le x \le 2, \end{cases}$ then

(I) $\lim_{x \to 1^-} f(x) = 1$

(II) $\lim_{x \to 1^+} f(x) = 2$

(III) $\lim_{x \to 1} f(x) = 2$

(IV) $\lim_{x \to 1} f(x) = 1$

Of these statements

(i) I and III are correct

(ii) II and IV are correct

(iii) I and II are correct

(iv) III alone is correct

(Choose the correct answer)

(h) Find the maximum value of $\sin x + \cos x$.

(i) Evaluate : $\lim_{x\to 0} \frac{1-\cos x}{3x^2}$

(j) The value of 'C' in Lagrange's Mean Value theorem for $f(x) = \alpha x^2 + \beta x + \gamma$, $\alpha \neq 0$ in $[a, b]$ is given by

(i) $\frac{a+b}{2}$ (ii) $\sqrt{ab}$

(iii) $\frac{2ab}{a+b}$ (iv) $\frac{a}{b}+\frac{b}{a}$

(Choose the correct answer)

2. Answer the following questions : $2\times5=10$

(a) Show that the following set

$\left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3} - \ldots\ldots\right\}$ is closed

but not open.

(b) Show that the series $\sum \frac{(-1)^{n+1}}{n^p}$ is absolutely convergent for $p > 1$, but conditionally convergent for $0 < p \leq 1$.

(c) Examine the continuity at $x = 0$ of the function $f(x) = [x] - [-x]$ where $[x]$ denotes the largest integer $\leq x$.

(d) Verify Cauchy's Mean Value theorem for the functions $f(x) = \sin x$ and $g(x) = \cos x$ in $\left[-\frac{\pi}{2}, 0\right]$.

(e) Use Taylor's theorem to show that

$\cos x \geq 1 - \frac{x^2}{2}$ for all $x \geq 0$.

3. Answer any *four* parts : $5\times4=20$

(a) Prove that a set is closed if and only if its complement is open. 5

(b) Show that

(i) $\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+\ldots+\frac{1}{(2n)^2}\right]=0$

(ii) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ 5

(c) State the Leibnitz test for convergence of an alternating series. Applying the test show that the series

$$1-\frac{1}{3\cdot 2^2}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 4^2}+\cdots\cdots \text{ is convergent.}$$

1+4=5

(d) Show that the series is convergent 5

$$\frac{1\cdot 2}{3^2\cdot 4^2}+\frac{3\cdot 4}{5^2\cdot 6^2}+\frac{5\cdot 6}{7^2\cdot 8^2}+\cdots\cdots$$

(Use Comparison test)

(e) If $f(x)=\begin{cases}\dfrac{x\left(e^{\frac{1}{x}}-e^{-\frac{1}{x}}\right)}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}, & x\neq 0\\ 0, & x=0\end{cases}$

show that f is continuous but not derivable at $x=0$. 5

(f) (i) Find the values of a and b in order that

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$$

(ii) Evaluate : $\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)$ 3+2=5

4. Answer either (a) and (b) or (c) and (d) : 5×2=10

(a) Prove that every infinite bounded set has a limit point.
Can an infinite unbounded set have a limit point ? Justify your answer. 4+1=5

(b) State Cauchy's General Principle of convergence of a sequence. Using this show that the following sequence $\{S_n\}$ where

$$S_n=1+\frac{1}{2!}+\frac{1}{3!}+\cdots\cdots+\frac{1}{n!} \text{ is convergent.}$$

1+4=5

(c) Prove that a convergent sequence of real numbers is bounded. Is the converse true ? Justify your answer. 3+2=5

(d) Show that the sequence $\{S_n\}$ defined by recursion formula $S_1=\sqrt{2},\ S_{n+1}=\sqrt{2S_n}$ converges to 2. 5

5. Answer either (a) and (b) or (c) and (d) : 5×2=10

(a) Prove that a necessary condition for convergence of an infinite series $\sum_{n=1}^{\infty}u_n$ is $\lim_{n\to\infty}u_n=0$.

With an example show that it is not a sufficient condition. 3+2=5

(b) State Raabe's test for convergence of a series. Applying this test, examine the convergence of the series 1+4=5

$$1+\frac{3}{7}x+\frac{3}{7}\cdot\frac{6}{10}x^2+\frac{3}{7}\cdot\frac{6}{10}\cdot\frac{9}{13}x^3+\ldots\ldots$$

(c) Applying Logarithmic test prove that the series $1+\frac{1!}{2}x+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\ldots\ldots..$ converges if $x<e$ and diverges if $x \geq e$. 5

(d) Test the convergence of the series

$$1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2+$$

$$\frac{\alpha(\alpha+1)(\alpha+2)\,\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3+\ldots\ldots$$

for all positive values of x ; α, β, γ being all positive. 5

6. Answer the following questions :

(a) Prove that if a function is continuous in a closed interval, then it is bounded therein. 5

Or

Show that the function f defined by

$$f(x)=\begin{cases}-x, & \text{if } x \text{ is rational}\\ x, & \text{if } x \text{ is irrational}\end{cases}$$

is continuous only at $x = 0$. 5

(b) Define uniform continuity of a function on an interval.

Prove that every uniformly continuous function on an interval is continuous on that interval.

Justify with an example that the converse is not true. 1+2+2=5

Or

(c) Show that the function $f(x)=\frac{1}{x^2}$ is uniformly continuous on (a, ∞), where $a>0$ but not uniformly continuous on $(0, \infty)$. 5

7. Answer any *two* parts :

(a) State Rolle's theorem. Using it prove that if $f'(x)$ and $g'(x)$ exist for all $x \in [a, b]$, and $g'(x) \neq 0$ for all $x \in (a, b)$, then for some c between a and b, $\frac{f(c)-f(a)}{g(b)-g(c)}=\frac{f'(c)}{g'(c)}$.

1+4=5

(b) Show that

$$\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}, \text{ if } 0<u<v.$$

Hence deduce that $\frac{\pi}{4}+\frac{3}{25}<\tan^{-1}\frac{4}{3}<\frac{\pi}{4}+\frac{1}{6}$.

3+2=5

(c) Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is $\frac{2a}{\sqrt{3}}$. 5

(d) Find Maclaurin's power series expansion for the function

$f(x) = \log(1+x)$ for $-1 < x \leq 1$. 5