2014

MATHEMATICS

(Major)

Paper : 3.2

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. Answer the following as directed : 1×6=6

(a) Let W be a subset of the vector space $\mathbb{R}^3(\mathbb{R})$ defined by

$$W = \{(x, y, z) \mid x, y, z \in \mathbb{R} \text{ and } x + 2y + 4z = 4\}$$

Is W a subspace? Justify.

(b) Show that each non-zero singleton set $\{x\}$ of a vector space is linearly independent.

(c) If M is the vector space of all $m \times n$ matrices, determine dim M.

(d) The product of all the eigenvalues of a square matrix A is equal to

(i) 0
(ii) 1
(iii) $|A|$
(iv) $\frac{1}{|A|}$

(Choose the correct answer)

(e) The matrix

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$$

satisfies the equation

(i) $A^2 + 5A + 7I = 0$

(ii) $A^2 + 5A - 7I = 0$

(iii) $A^2 - 5A - 7I = 0$

(iv) $A^2 - 5A + 7I = 0$

(Choose the correct answer)

(f) State the condition for a system of n equations in n unknowns to have a unique solution.

2. Answer the following : 2×2=4

(a) Show that the union of two subspaces of a vector space is not necessarily a subspace of the vector space.

(b) Show that the matrices A and A' have the same eigenvalues.

3. Answer any *one* part : 10

(a) (i) Let U and W be subspaces of a vector space $V(F)$. Then show that $V = U \oplus W \Leftrightarrow V = U + W$ and $U \cap W = \{0\}$.

(ii) Show that the vectors (1, 2, 5), (2, 5, 1) and (1, 5, 2) are linearly independent in the vector space $\mathbb{R}^3(\mathbb{R})$.

(iii) If V is a finite dimensional vector space and $\{v_1, v_2, \cdots, v_n\}$ is a linearly independent subset of V, then prove that it can be extended to form a basis of V. 4+2+4=10

(b) (i) If W_1 and W_2 are subspaces of a finite dimensional vector space $V(F)$, then show that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

(ii) Define basis of a vector space. Determine whether or not the vectors (1, 1, 2), (1, 2, 5), (5, 3, 4) form a basis of the vector space $\mathbb{R}^3(\mathbb{R})$. 5+5=10

4. Answer any *two* parts : 5×2=10

(a) Define rank and nullity of a linear transformation T from a vector space $V(F)$ to a vector space $W(F)$. Find rank and nullity of the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (x+y+z, 2x+2y+2z)$$ 1+4=5

(b) Let T be a linear operator on a vector space $V(F)$ and rank T^2 = rank T. Then show that range $T \cap \ker T = \{0\}$. 5

(c) Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be a linear mapping defined by

$$T(x, y, z) = (3x+2y-4z, x-5y+3z)$$

Find the matrix A representing T relative to the bases $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ of $\mathbb{R}^3(\mathbb{R})$ and $B' = \{(1, 3), (2, 5)\}$ of $\mathbb{R}^2(\mathbb{R})$. 5

5. Answer any *one* part : 10

(a) (i) Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$$

(ii) State the Cayley-Hamilton theorem. Verify it for the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$$

Hence find A^{-1}. 4+6=10

(b) (i) Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that A and B have different characteristic polynomials but have the same minimal polynomial.

(ii) Show that the system of equations

$$3x+y+z=8$$
$$-x+y-2z=-5$$
$$2x+2y+2z=12$$
$$-2x+2y-3z=-7$$

is consistent and hence solve them. 5+5=10

GROUP—B

(Vector)

(*Marks* : 40)

6. Answer the following as directed : 1×4=4

(a) Evaluate $(\vec{r}\cdot\hat{i})\hat{i}+(\vec{r}\cdot\hat{j})\hat{j}+(\vec{r}\cdot\hat{k})\hat{k}$.

(b) Write the value of $\text{div}(\nabla\phi\times\nabla\psi)$.

(c) If ϕ is a continuously differentiable scalar point function, the value of curl grad ϕ is

(i) 1

(ii) -1

(iii) $\vec{0}$

(iv) None of the above

(Choose the correct answer)

(d) If C is a closed curve, find $\oint_C \vec{r}\cdot d\vec{r}$.

7. Answer the following : 2×3=6

(a) Prove the identity

$$\vec{a}\times[\vec{a}\times(\vec{a}\times\vec{b})]=(\vec{a}\cdot\vec{a})(\vec{b}\times\vec{a})$$

(b) Find div curl $\vec{F}$, if $\vec{F}=x^2y\hat{i}+xz\hat{j}+2yz\hat{k}$.

(c) Interpret the relations

$$\vec{r}\cdot\frac{d\vec{r}}{dt}=0 \text{ and } \vec{r}\times\frac{d\vec{r}}{dt}=\vec{0}$$

8. Answer any *one* part : 10

(a) (i) Prove that

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})+(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})=\vec{0}$$

(ii) Prove that $\nabla^2(\frac{1}{r})=0$, where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$.

(iii) If

$$\vec{A}=xyz\hat{i}+xz^2\hat{j}-y^3\hat{k}$$

and

$$\vec{B}=x^3\hat{i}-xyz\hat{j}+x^2z\hat{k}$$

calculate

$$\frac{\partial^2\vec{A}}{\partial y^2}\times\frac{\partial^2\vec{A}}{\partial x^2}$$

at the point (1, 1, 3). 3+4+3=10

(b) (i) Prove that

$$[\vec{b}+\vec{c}\;\;\vec{c}+\vec{a}\;\;\vec{a}+\vec{b}]=2[\vec{a}\;\vec{b}\;\vec{c}]$$

Hence show that $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar, iff $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$, $\vec{a}+\vec{b}$ are coplanar.

(ii) Prove that

$$\text{curl}(f\vec{F}) = \text{grad}\, f \times \vec{F} + f(\text{curl}\, \vec{F})$$

where f is a scalar point function.

(iii) Find the unit normal vector to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$. 4+3+3=10

9. Answer any *two* parts : 5×2=10

(a) If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, then prove that

$$[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$ 5

(b) Prove that

$$\text{curl}(\vec{a}\times\vec{b}) = (\vec{b}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{b} + \vec{a}\,\text{div}\,\vec{b} - \vec{b}\,\text{div}\,\vec{a}$$ 5

(c) Show that

$$\int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt = \vec{r}\times\frac{d\vec{r}}{dt} + C$$

where C is an arbitrary constant vector. If $\vec{r}(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, then prove that

$$\int_1^2\left(\vec{r}\times\frac{d^2r}{dt^2}\right)dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$$

2+3=5

10. Answer any *one* part : 10

(a) (i) Evaluate $\int_V (2x+y)\,dV$, where V is a closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0,\ y = 0,\ y = 2$ and $z = 0$.

(ii) Verify Green's theorem in the plane for

$$\oint_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$$

where C is the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$ and $(0, 2)$.

5+5=10

(b) (i) Find the work done in moving a particle once around a circle C in the xy-plane, if the circle has centre at the region and radius 2 and, if the force field $\vec{F}$ is given by

$$\vec{F} = (2x - y + 2z)\hat{i} + (x + y - z)\hat{j} + (3x - 2y - 5z)\hat{k}$$

(ii) State Gauss' divergence theorem. Using it, evaluate $\iint_S \vec{F}\cdot\hat{n}$, where $\vec{F} = x\hat{i} - y\hat{j} + (z^2 - 1)\hat{k}$ and S is the cylinder formed by the surfaces $z = 0,\ z = 1,\ x^2 + y^2 = 4$. 5+5=10

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