2014

MATHEMATICS

(Major)

Paper : 3.1

(Abstract Algebra)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : 1×10=10

(a) Let G and G' be finite groups such that $\gcd(o(G), o(G')) = 1$. Define a homomorphism from G to G'.

(b) State the fundamental theorem of group homomorphism.

(c) Let $f : G \to G'$ be a group homomorphism. Let $a \in G$ be such that $o(a) = n$ and $o(f(a)) = m$. Then $o(f(a)) / o(a)$ and f is one-one if and only if

(i) $m > n$

(ii) $m < n$

(iii) $m = n$

(iv) $m = n = 1$

(Choose the correct option)

(d) Let $R = \{0, 1, 2\} \bmod 3$. What is the characteristic of R?

(e) State whether True or False :

If an element a of a group G has only two conjugates in G, then $N(a)$ is a normal subgroup of G.

($N(a)$: normalizer of a in G)

(f) State Cauchy's theorem for a finite group G.

(g) If T is an automorphism of a group G, then $o(Ta) = o(a)$ for all $a \in G$. Now, for all $a, b \in G$

(i) $o(bab^{-1}) = o(b)$

(ii) $o(bab^{-1}) = o(a)$

(iii) $o(bab^{-1}) = o(Tb)$

(iv) $o(bab^{-1}) = 2$

(Choose the correct option)

(h) Give an example of a Euclidean domain.

(i) Let $R[x]$ be the ring of polynomials of a ring R and let

$$f(x) = a_0 + a_1 x + \cdots + a_m x^m$$

$$\text{and} \quad g(x) = b_0 + b_1 x + \cdots + b_n x^n$$

If $f(x) + g(x) \neq 0$, then

(i) $\deg(f(x) + g(x)) \leq \max(m, n)$

(ii) $\deg(f(x) + g(x)) \geq m + n$

(iii) $\deg(f(x) + g(x)) = m + n$

(iv) $\deg(f(x) + g(x)) \geq \max(m, n)$

(Choose the correct option)

(j) State True or False :

In a principal ideal domain, every non-zero prime ideal is maximal.

2. Answer the following questions : 2×5=10

(a) Let $\mathbb{Z}$ be the additive group of integers and $\phi : \mathbb{Z} \to \mathbb{Z}$ be defined by $\phi(x) = x + 1$, $x \in \mathbb{Z}$. Examine if ϕ is a homomorphism.

(b) If R is a ring with no non-zero nilpotent elements, then show that for any idempotent e, $ex = xe \; \forall \; x \in R$.

(c) The sum of two subspaces of a vector space is again a subspace.

Justify whether it is true or false.

(d) Let G be a group and $Z(G)$ be the centre of G. Show that if $\text{cl}(a) = \{a\}$, then $a \in Z(G)$. ($\text{cl}(a)$: the conjugacy class of a)

(e) Let f be a homomorphism from a ring R onto a ring R'. If e is the unity of R, then $f(e)$ is the unity of R'.

Justify whether this statement is true or false.

3. Answer the following questions : 5×4=20

(a) Let $G = (\mathbb{R}, +)$, $G' = (\{Z \in \mathbb{C} : |Z| = 1\}, \cdot)$ and $\phi : G \to G'$ is defined by

$$\phi(x) = \cos 2\pi x + i \sin 2\pi x, \quad x \in \mathbb{R}$$

Prove that ϕ is a homomorphism and determine $\ker \phi$.

Or

If $f : G \xrightarrow{\text{onto}} G'$ is a group homomorphism, prove that H is a normal subgroup of G if and only if $f(H)$ is a normal subgroup of G'.

(b) If R is a division ring, then show that the centre $Z(R)$ of R is a field.

Or

Let R be a ring having more than one element such that $aR = R$, for all $0 \neq a \in R$. Show that R is a division ring.

(c) Prove that a group of order p^2, where p is prime, is Abelian.

(d) Prove that every ideal in a Euclidean domain is a principal ideal.

4. Answer the following questions : 10×4=40

(a) Let H and K be two normal subgroups of a group G such that $H \subseteq K$. Prove that

$$\frac{G}{K} \cong \frac{G/H}{K/H}$$ 10

Or

Let G be the additive group of reals and N be the subgroup of G consisting of integers. Prove that $\frac{G}{N}$ is isomorphic to the group H of all complex numbers of absolute value 1 under multiplication.

(b) Let A, B, C be ideals of a ring R such that $B \subseteq A$. Show that

$$A \cap (B + C) = (A \cap B) + (A \cap C) = B + (A \cap C)$$ 10

Or

Let R be a commutative ring with unity. Show that every maximal ideal of R is also a prime ideal. Moreover, prove that if every ideal of R is prime, then R is a field. 3+7=10 *(c)* State Sylow's 1st and 3rd theorems for a group G. Let $o(G) = pq$, where p, q are distinct primes such that $p < q$, $p \nmid q-1$. Show that G is cyclic. 2+8=10

Or

Let G be a finite group and $a \in G$. Prove that

$$o(\mathrm{cl}(a)) = \frac{o(G)}{o(N(a))}$$ 10

(d) Show that $\mathbb{Z}[\sqrt{2}] = \{a + \sqrt{2}b : a, b \in \mathbb{Z}\}$ is a Euclidean domain. 10

Or

Prove that any ring can be imbedded into a ring with unity.

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