2014

MATHEMATICS

(Major)

Paper : 1.1

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

1. Choose the correct answer : 1×10=10

(a) Which of the following relations is symmetric but neither reflexive nor transitive?

(i) $R_1 = \{(a, b) \mid a^2 + b^2 = 1\}$ on the set of real numbers

(ii) $R_2 = \{(a, b) \mid (a - b)$ is divisible by $3\}$ on the set of all integers

(iii) $R_3 = \{(a, b) \mid a$ is a multiple of $b\}$ on the set of natural numbers

(iv) $R_4 = \{(a, b) \mid 1 + ab > 0\}$ on the set of real numbers

(b) If a and b are any two elements of a group G, then $(a^{-1}b)^{-1}$ is

(i) $a^{-1}b^{-1}$

(ii) ab^{-1}

(iii) $b^{-1}a$

(iv) $b^{-1}a^{-1}$

(c) If a is an element of a group G, then which of the following is not true?

(i) $o(a) = o(a^{-1})$

(ii) $o(a) < o(a^2)$

(iii) $o(a) \geq o(a^2)$

(iv) $o(a) = o(x^{-1}ax)$, $x \in G$

(d) The value of $\sin(\log i^i)$ is

(i) 1

(ii) −1

(iii) $\frac{1}{2}$

(iv) $\frac{1}{\sqrt{2}}$

(e) If $(1+x)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$, then $a_0 - a_2 + a_4 - \cdots$ is

(i) $2^{\frac{n}{2}} \sin\frac{n\pi}{4}$

(ii) $2^{\frac{n}{2}} \cos\frac{n\pi}{4}$

(iii) $2^{\frac{n+1}{2}} \cos\frac{n\pi}{4}$

(iv) $2^{\frac{n-1}{2}} \sin\frac{n\pi}{4}$

(f) Which one of the following is true?

(i) $\log_e(-\alpha + i\beta) = \frac{1}{2}\log_e(\alpha^2 + \beta^2) + i\tan^{-1}\frac{\alpha}{\beta}$

(ii) $\log_e(\alpha - i\beta) = \frac{1}{2}\log_e(\alpha^2 + \beta^2) - i\tan^{-1}\frac{\alpha}{\beta}$

(iii) $\log_e(\alpha - i\beta) = \frac{1}{2}\log_e(\alpha^2 + \beta^2) - i\tan^{-1}\frac{\beta}{\alpha}$

(iv) $\log_e(\alpha + i\beta) = \frac{1}{2}\log_e(\alpha^2 + \beta^2) - i\tan^{-1}\frac{\beta}{\alpha}$

(g) If α, β, γ, δ are the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$, then the value of $\Sigma\alpha^2\beta$ is

(i) $pr - 4s$

(ii) $p^2 - 2q$

(iii) $-pq + 3r$

(iv) $r^2 - 2qs$

(h) If A is a symmetric matrix, then $B'AB$, where B is any square matrix of same order, is

(i) symmetric

(ii) skew-symmetric

(iii) Hermitian

(iv) skew-Hermitian

(i) If the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear, then the rank of the matrix

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

is

(i) 3

(ii) < 3

(iii) > 3

(iv) ≥ 3

(j) If the number of variables in a system of non-homogeneous equations $AX = B$ is n, then the system possesses a unique solution, if

(i) rank $A <$ rank $[A\ B]$

(ii) rank $A >$ rank $[A\ B]$

(iii) rank $A =$ rank $[A\ B] = n$

(iv) None of the above

SECTION—B

2. Give answers to the following questions : $2 \times 5 = 10$

(a) With an example, show that we can have maps f and g such that $g \circ f$ is one-one and onto but f need not be onto and g need not be one-one.

(b) If A and B are symmetric matrices, then show that AB is symmetric if and only if $AB = BA$.

(c) Prove that every cyclic group is Abelian.

(d) Expand $\sin^3 x$ in ascending powers of x.

(e) Solve the equation

$$2x^4 - 3x^3 - 3x^2 - 3x - 1 = 0$$

which has one root $1+\sqrt{2}$.

SECTION—C

3. Answer any *four* parts : 5×4=20

(a) Let G be a group. For any $a \in G$, define equivalence class $\mathrm{Cl}(a)$ of a. If ~ is an equivalence relation on a non-empty set X, then prove that for any $a, b \in X$

(i) $\mathrm{Cl}(a) \neq \phi$

(ii) either $\mathrm{Cl}(a) \cap \mathrm{Cl}(b) = \phi$ or $\mathrm{Cl}(a) = \mathrm{Cl}(b)$

(iii) $X = \bigcup_{a \in X} \mathrm{Cl}(a)$ 1+4=5

(b) Prove that a mapping $f : X \to Y$ is bijective if and only if $\exists$ a mapping $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are identity maps on X and Y respectively. 5

(c) Show that the union of two subgroups of a group G is a subgroup if and only if one of them is contained in the other. 5

(d) (i) If

$$\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$$

then prove that

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$$

and

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$$

(ii) Prove that

$$(-i)^{(-i)} = e^{(4n-1)\frac{\pi}{2}}$$ 3+2=5

(e) If α, β, γ are the roots of the equation $x^3 + ax + b = 0$, find the equation whose roots are $\alpha^2 + \beta^2, \beta^2 + \gamma^2, \gamma^2 + \alpha^2$. 5

(f) If A is a non-singular matrix, then show that adj adj $A = |A|^{n-2} A$ 5

SECTION—D

4. Answer any *one* part : 10

(a) (i) Suppose for fixed $n > 1$ and for all a, b in a group G, $(ab)^n = a^n b^n$. Show that $(ab)^{n-1} = b^{n-1} a^{n-1}$.

(ii) Prove that an infinite cyclic group has precisely two generators.

(iii) Prove that a subgroup H of a group G is a normal subgroup in G if and only if $g^{-1}hg \in H$ for all $h \in H$, $g \in G$. 2+4+4=10

(b) *(i)* Show that a subgroup of index 2 in a group G is a normal subgroup of G.

(ii) If H is a subgroup of a group G and $a \in G$, then prove that

$$Ha = H \Leftrightarrow a \in H$$

(iii) Let a, $n (n \geq 1)$ be any integers such that g.c.d. $(a, n) = 1$, then prove that

$$a^{\phi(n)} \equiv 1 \pmod{n}$$ 3+2+5=10

5. Answer any *one* part : 10

(a) *(i)* If z is a complex number such that the amplitude of $\frac{z-i}{z+i}$ is $\frac{\pi}{4}$, then show that the point z lies on a circle in the complex plane.

(ii) If $x < (\sqrt{2}-1)$, then prove that

$$2\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots\right) =$$

$$\frac{2x}{1-x^2} - \frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3 + \frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5 - \dots$$

(iii) If $\tan(\alpha + i\beta) = x + iy$, then prove that $x^2 + y^2 + 2x\cot 2\alpha - 1 = 0$ and $x^2 + y^2 - 2y\coth 2\beta + 1 = 0$. 3+3+4=10

(b) *(i)* If a and b are complex numbers, then show that

$$|a + \sqrt{a^2 - b^2}| + |a - \sqrt{a^2 - b^2}| = |a + b| + |a - b|$$

(ii) Separate into its real part and imaginary part of the expression $(\alpha + i\beta)^{x+iy}$.

(iii) Prove that

$$(1 + \cosh\theta + \sinh\theta)^n = 2^n \cosh^n \frac{\theta}{2}\left(\cosh\frac{n\theta}{2} + \sinh\frac{n\theta}{2}\right)$$

3+4+3=10

6. Answer any *two* parts : 5×2=10

(a) If α, β, γ are roots of the equation $x^3 - px^2 + qx - r = 0$, find the value of $\sum \frac{\beta}{\gamma} + \frac{\gamma}{\beta}$ and $\sum \alpha^3\beta^3$. 5

(b) *(i)* Find the equation whose roots are the roots of the equation $x^3 + 3x^2 - 8x + 1 = 0$, each increased by 1.

(ii) Apply Descartes' rule of signs to find the nature of roots of the equation $x^6 - 3x^2 - 2x - 3 = 0$. 2+3=5

(c) If the equation $x^3 + px^2 + qx + r = 0$ has a root $\alpha + i\alpha$, where p, q, r and α are reals, then prove that $(p^2 - 2q)(q^2 - 2pr) = r^2$. 5

(d) Solve the following equation by Cardon's method $x^3 + 9x^2 + 15x - 25 = 0$. 5

7. Answer any *two* parts : 5×2=10

(a) Define a Hermitian matrix. Prove that if A is an n-rowed square matrix, then $\overline{A}$ is Hermitian or skew-Hermitian according as A is Hermitian or skew-Hermitian. 1+4=5

(b) Let A and B be two n-rowed invertible matrices. Show that AB is also an n-rowed invertible matrix and $(AB)^{-1} = B^{-1}A^{-1}$. 5

(c) Reduce the matrix to the normal form and hence determine its rank

$$\begin{pmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{pmatrix}$$

4+1=5

(d) Show that the system of equations

$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$x + 4y + 7z = 30$$

is consistent and solve them. 5

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